New Algorithms to Find Reliability and Unreliability Functions of the Consecutive \( k \)-out-of-\( n \): F Linear & Circular System

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Abstract: The consecutive \( k \)-out-of-\( n \) system consists of \( n \) components. The system fails if at least \( k \) consecutive components are in the failure state. In this paper, new algorithms to find the reliability and unreliability functions of the consecutive \( k \)-out-of-\( n \) system are obtained in this context, the Index Structure Function (ISF), and the equivalence relations are defined to partition the failure and the functioning space of the consecutive \( k \)-out-of-\( n \): F system into finite mutual pairwise disjoint classes respectively, where the reliability and unreliability functions are the summations of reliability and unreliability of these equivalence classes. For the linear \( k \)-out-of-\( n \) system, the boundary conditions are given to omit a specified failure states from the failure space of the circular case to achieve the failure states of the linear case, which facilitate the evaluation of the reliability and unreliability functions of the consecutive \( k \)-out-of-\( n \) system.

Keywords: Consecutive \( k \)-out-of-\( n \) system, System reliability, equivalence relation, modular arithmetic.

Notations:

- \( L(C) \): Linear (Circular)
- \( i.i.d. \): Independent and Identically distributed
- \( \text{mod} \): Modular arithmetic function
- \( \text{gcd} \): greatest common divisor
- \( I'_j \): \( \{i,i+1,...,j\} \)
- \( P(I'_j) \): The power set of \( I'_j \)
- \( X = \{x_1, <...<x_i\} \): a set \( X = \{x_1, x_2, ..., x_j\} \) such that \( x_i < x_j \) for all \( i < h \)
- \( i \oplus s \): \( (i+s) \text{mod} j \), if \( i+s = nj \Rightarrow i \oplus s = j \)
- \( p_i(q_j) \): The reliability (unreliability) of the \( i \)-th component.
- \( \bar{X} \): The complement of the set \( X \).
- \( P_x = \prod_{j \in X} p_j \prod_{j \notin X} q_j \)
- \( P'_x = p(x) = p^{r}q' \)
- \( R(X) = P_x \), \( F(X) = p_x \)
- \( R[X] \): Reliability of the class \( [X] \)
- \( F[X] \): Unreliability of the class \( [X] \)
- \( C(k,n) \): Consecutive \( k \)-out-of-\( n \) system.
- \( CL(k,n) \): Circular \( k \)-out-of-\( n \) system.
- \( CC(k,n) \): Circular \( k \)-out-of-\( n \) system.
- \( |X| \): The cardinality of the set \( X \).
- \( f^n \): The composite function \( f \) \( n \) times.
- \( R^{(c)}_i, F^{(c)}_i \): The reliability, unreliability function of the consecutive \( k \)-out-of-\( n \) system for linear (circular) when the number of failed components is \( j \).

1. Introduction

Through the last 2 decades in the last century, many researchers studied intensively and extensively the reliability of the consecutive \( k \)-out-of-\( n \) system for various applications, low expense and high reliability, the system consists of \( n \) components, the components are ordered sequentially in a line or a circle, the system fails if, and only if at least \( k \) consecutive components are in the failure state. Chiang and Niu [5] provided the two famous applications (Telecommunication System with \( n \) Relay Stations (either satellites or ground stations) and Pipeline of Transmit Oil System with \( n \) Pumps), Microwave Stations of a Telecom Network [4], Vacuum System in an Electron Accelerator and Photographing of a Nuclear Accelerator [9].

Chiang & Niu [5] obtained the first recursive algorithm for computing the reliability of the consecutive \( k \)-out-of-\( n \) system “\( CL(k,n) \)”, Bollinger [2-3] introduced a simple and direct combinatorial formula for calculating the failure function of the system, while Derman et al. [6] was the first one who introduced and calculated the reliability of the consecutive \( k \)-out-of-\( n \) system “\( CC(k,n) \)”, they provided a recursive algorithm for computing the reliability of the \( CL(k,n) \) and \( CC(k,n) \) with \( i.i.d. \) components, Shanthikumar [13] computed the reliability of the \( CL(k,n) \), when the components are stochastically independent with unequal failure probabilities, Hwang [7] studied both the \( CL(k,n) \) & \( CC(k,n) \), two recursive algorithms were introduced using two arguments for the \( CL(k,n) \), he extended also the argument of Derman et al. [6] to introduce a recursive equation for the \( CC(k,n) \). Lambiris and Papastavridis [10] introduced the first exact formula for the \( CL(k,n) \) and \( CC(k,n) \), where the components are \( i.i.d. \), Antopoulou and Papastavridis [1] studied only the \( CC(k,n) \); they introduced a faster recursive algorithm than Hwang [7] and all other previous algorithms that obtaining the reliability of the stochastically independent components with fewer components, Wu & Chen [15] generalized the system through adding some conditions and restrictions, and transformed the \( CC(k,n) \) to the \( CL(k,n) \) by adding virtual components to the circular system \( n+1, n+2, ... \).
... n+i, ..., n+k-1, where n+i=i. Many recursive algorithms, lower and upper bounds, optimal design and exact formula were introduced by [8-9], [11], [12] and [14].

In this paper new algorithms to find the reliability and unreliability functions of the CC(k,n) & CL(k,n) are obtained. We recall and adapt the needed facts, and results from the set theory that is concerning relation between components in the circular system, the ISF and equivalence relations are defined, these relations are extended to represent the failure and functioning space of the CC(k,n) & CL(k,n), which yields to compute the reliability and unreliability functions.

The following assumptions are assumed to be satisfied by all systems below:
1. The state of the component and the system is either “functioning” or “failed”
2. All the components are mutually statistically independent.

2. The Index Structure Function

Definition 2.1: Consider a system consists of n stochastically independent components, I_i is the indices of the components. Define the Index Structure Function (ISF) of the i-th component

Γ(i) = {0, if the i-th component functions
1, if the i-th component fails

The set X = {i ∈ I_i : Γ(i) = 1} ⊆ I_i, represents the system and consists of all indices of the failed components e.g., the set X = {1,2}, for simply, X = 12 indicates that the 1-st and the 2-nd components are only the failed components. Define the ISF of the whole system, I : P(I_i) → {0,1},

I(X) = 0, if the system is functioning
1, if the system is failed

Where, P(I_i) is the failure space of the components. The ISF is concerning the components positions and indices that is determining whether the system fails or not.

Definition 2.2: X ⊆ I_i is called functioning set, if I(X) = 0, vice versa, a failed set if I(X) = 1. In this context, the ISF divides P(I_i) into two main following disjoint sub collections:
- Θ = {X ∈ P(I_i) : I(X) = 0} the functioning space of the system
- Ψ = {Y ∈ P(I_i) : I(Y) = 1} the failed space of the system.

3. Reliability Using the ISF

Definition 3.1: The reliability of the i-th component is P(Γ(i) = 0) = p_i, hence if the system represented by the functioning set X ∈ Θ, then the reliability of the system is R(X) = P(I(X) = 0) = p_x, which implies, the reliability function of the system R is the probability that the system has any functioning set; i.e., R = ∑_X∈Θ R(X).

While, the unreliability of the i-th component is P(Γ(i) = 1) = q_i = 1−p_i. If the system represented by the failed set Y ∈ Ψ, then the system unreliability is F(Y) = P(I(Y) = 1) = p_y and the unreliability function of the system F is the probability that the system has any failed set F = ∑_Y∈Ψ F(Y) = 1−R.

4. The Circular System Model

The circular system model is any system consists of n stochastically independent components, and arranged in a circle. Define the circular system model using ISF, where I_i is the indices of the components, and the set X ⊆ I_i represents the system, which consists of all failed components. The positions of the components in the circle applies that the participation of any component in the system status, “whether functioning or failed” is equivalent to the participation of any other component. To simulate this property, define the bijection function, f_n : I_i → I_i, f_n(x) = (x mod n) + 1 for any x ∈ I_i, and a relation between any X, Y ∈ P(I_i) such that X = Y if, and only if, there exist α ∈ Z such that f_n^α(X) = Y.

Lemma 4.1: (≡) is an equivalence relation.

Proof: Reflexivity: X ~ X since X = f_n^0(X)
Transitivity: If X ~ Y then exists α_i such that Y = f_n^α(X) and Y ~ Z then there exists α_j such that Z = f_n^α(Y) ⇒ Z = f_n^α(f_n^α(X)) = f_n^α+α_j(X) ⇒ X ~ Z.
Symmetry: If X ~ Y then exists α such that Y = f_n^α(X), hence X = f_n^-α(Y) ⇔ Y ~ X.

According to lemma 4.1 P(I_i) is a union of a finite partition of mutually disjoint classes like [X] = {f_n^α(X) : α = 1,2,...,n}, as well as the Θ and Ψ.

For example, in the circular system of 6 components, [13] = {13,24,35,46,15,26}.

4.1. The Rotations in the Circular System

Definition 4.1.1: Let X ∈ P(I_i) represents the circular system, define the orbit of X as β_x = min {α ∈ Z : f_n^α(X) = X}. Note that [X] = {f_n^α(X) : α = 1,2,...,β_x}. 
Lemma 4.1.1: For all \( X \in P(T^*_n) \), \( \beta_x \) divides \( n \).

**Proof:** Since \( f^n_\alpha(X) = X \) for all \( X \in P(T^*_1) \), then \( \beta_x \leq n \). If \( \beta_x = n \) we done. Otherwise if \( \beta_x < n \), and \( \beta_x \) does not divide \( n \), then \( n = a \beta_x + b \), where \( 0 < b = n \mod \beta_x \).

\[ X = f^n_\alpha(X) = f^n_{\beta_x+b}(X) = f^n_{\beta_x}(f^n_b(X)) = f^n_b(X) \] but, \( b < \beta_x \), which contradicts that \( \beta_x \) is the minimum integer such that \( X = f^n_{\beta_x}(X) \).

**Note:** If \( Y \in [X] \)

1. There exist \( \alpha \in \mathbb{Z} \) such that \( f^n_\alpha(X) = Y \), but \( f^n_{\beta_x}(Y) = f^n_{\beta_x}(f^n_\alpha(X)) = f^n_\alpha(f^n_{\beta_x}(X)) = f^n_\alpha(X) = Y \)

\[ \Rightarrow \beta_y = \beta_x \]

2. \( |X| = |Y| \).

3. \( R[X] = \sum_{z \in [X]} R(z) = \sum_{z \in [X]} p_z = \sum_{z \in [X]} p_{f^n_{\beta_x}(z)} \).

4. \( F[X] = \sum_{z \in [X]} F(z) = \sum_{z \in [X]} p_z = \sum_{z \in [X]} p_{f^n_{\beta_x}(z)} \).

**Definition 4.1.2:** Let \( X = \{x_1, \ldots, x_j\} \subseteq T^*_1 \), represents the circular system, define \( d_x = (d_{x1}, d_{x2}, \ldots, d_{xj}) \) the rotation of \( X \), where \( d_{x1} \geq 1 \) is the minimum integer number such that \( f^{d_{x1}}(x_i) = x_{i+1} \), for \( i = 1, 2, \ldots, j-1 \), and \( f^{d_{xj}}(x_j) = x_1 \), also define the following:

1. \( d_x = (d_{x1}, d_{x2}, \ldots, d_{xj}) \) is the \( p \)-th rotation of \( X \). (note that \( d_{x1} \geq 1 \)).

2. The set of all rotations of the set \( X \) is \( D(X) = \{d_x : r = 1, 2, \ldots, j\} \).

**Note:** If \( X = \{x_1, \ldots, x_j\} \subseteq T^*_1 \), then, anyone can easily shows that

1. \( n = \sum_{i=1}^j d_{x_i} \).

2. If \( d_{x_i} = t \) for all \( i = 1, 2, \ldots, j \), then \( \beta_x = t \).

**Definition 4.1.3:** Let \( X, Y \in P(T^*_1) \) be any states of the circular system, we say “\( Y \) is a rotation of \( X \)”, denoted by \( x \sim y \) if \( d_x \in D(X) \) or there exists \( r \in \{1, 2, \ldots, j\} \) such that \( d_x = d_{x_r} \).

**Lemma 4.1.3:** (\( \sim \)) is an equivalence relation.

**Proof:** Reflexivity: \( X \sim X \Rightarrow d_x = d_x \)

Transitivity: Let \( X \sim X \Leftrightarrow \) there exist \( r \) such that \( d_x = d_{x_r} \), and if \( Y \sim Z \Leftrightarrow \) there exist \( r \) such that \( d_z = d_{z_r} \), then \( d_x = d_{x_r} = d_{z_r} \).

Symmetry: \( X \sim Y \Leftrightarrow \) there exist \( r \) such that \( d_x = d_{x_r} \Rightarrow d_x = d_{x_r} \Rightarrow Y \sim X \).

**Theorem 4.1.1:** \( X = Y \Leftrightarrow X \sim Y \).

**Proof:** Assume \( X = Y \Leftrightarrow \) there exist \( \alpha \) such that \( f^n_\alpha(X) = Y \). \( f^n_\alpha \) is a bijection function for each \( \alpha \), hence, \( |X| = |Y| = j \). Let \( Y = \cup_{i=1}^j f^n_{\alpha_i}(x_i) \). WLOG, let \( f^n_{\alpha_i}(x_i) = y_i \) for all \( i = 1, 2, \ldots, j \), if \( d_x = (d_{x1}, d_{x2}, \ldots, d_{xj}) \), then \( f^n_{\alpha_i}(x_i) = f^n_{\alpha_i}(f^n_{\beta_x}(x_i)) = f^n_{\alpha_i}(d_{x_1}) = f^n_{\alpha_i}(d_{x_1}) \Rightarrow f^n_{\alpha_i}(x_i) \Rightarrow X \sim Y \).

Conversely: if \( X \sim Y \), there exist \( r \in [1, \ldots, j] \) such that \( d_x = d_{x_r} \). Since \( |X| = |Y| = j \), let \( X = \cup_{i=1}^j x_i, Y = \cup_{i=1}^j y_i \), as a result of \( r \)-th rotation, the element \( x_i \) will rotate to the element \( y_{f_j(r)} \), define the function \( g(x_i) = y_{f_j(r)} \), let \( \alpha = \min \{ \gamma \in \mathbb{Z} : f^n_{\alpha_i}(x_i) = g(x_i), \forall x_i \in X \} \), if and only if \( f^n_{\alpha_i}(x_i) = Y \Leftrightarrow \alpha = \gamma \).

For example in the circular system with 10 components, if \( X = \{149\}, Y = \{168\} \in P(T^*_1) \Rightarrow d_x = (3, 5, 2), d_y = (5, 2, 3) \Rightarrow d_y = d_{y_i} \Rightarrow g(x_i) = y_{f_j(r)} \Rightarrow g(x_i) = g(1) = y_1 = 8, g(x_2) = g(4) = y_1 = 1, g(x_3) = g(9) = y_2 = 6 \Rightarrow \alpha = \min \{ \gamma \in \mathbb{Z} : f^n_{\alpha_i}(x_i) = g(x_i), \forall x_i \in X \} = \min \{7, 17, \ldots \} \Rightarrow d_1 \in \{ \{149\} \} = \{168\}

**Note:** According Theorem 4.1.1 if \( d_x = d_{x_r} \Rightarrow |X| = |Y| \).

**Theorem 4.1.2:** Consider a circular system of \( n \) components, \( j \) the number of failed components, then

1. If \( a \in \mathbb{Z} \) such that \( a \times j = n \), then there exist \( X_a \in P(T^*_1) \) such that \( |X_a| = j \) and \( \beta_x = a \).

2. If \( X \in P(T^*_1) \) such that \( |X| = j \) and

\[ d_x = (d_{x_1}, d_{x_2}, \ldots, d_{x_{j-1}}, d_{x_j}, d_{x_{j-1}}, \ldots, d_{x_1}) \]

\[ \Rightarrow \beta_x = \sum_{i=1}^j d_{x_i} \].

3. If \( s, a \in \mathbb{Z} \) such that \( a \times j = n \), and \( s \) divide \( j \), then there exist \( Y_s \in P(T^*_1) \) and \( |Y_s| = j \) such that \( \beta_{y_s} = s \times a \).

**Proof:**

1. Take \( d_{x_i} = (a, a, \ldots) \), fix any \( x_i \in T^*_1 \), and \( x_{i+1} = f^n_{\alpha_i}(x_i) \) : \( i \in T^*_1 \), hence \( f^n_{\alpha_i}(x_i) = x_i \). Define

\[ X_a \in P(T^*_1) \] such that \( X_a = \cup_{i=1}^j x_i \), then
\[ f_n^e(x_n) = f_n^e\left( \bigcup_{i=1}^{a} X_i \right) = \bigcup_{i=1}^{a} f_n^e(x_i) = \bigcup_{i=0}^{a-1} x_{i+1} = X \]
\[ \Rightarrow \beta_x = a . \]

2. Use theorem 4.1.1 \( d_x = d_x^s \)
\[ \alpha = \min \{ y \in \mathbb{Z} : f_n^e(x_i) = x_f^{(y)} \} = \sum_{i=1}^{a} d_x^s = \beta_x \]

3. Use 1, there exist \( X_a \in P(1^n) \) with \( \beta_x = a \), where
\[ d_x = \left( a, a, \ldots, a \right) \text{ times} \], take \( d_x^s, d_x^s, \ldots, d_x^s \) not all of them \( a \), such that \( sa = \sum_{i=1}^{a} d_x^s \), define the set \( Y \) such that
\[ d_x = \left( d_x^s, d_x^s, \ldots, d_x^s, d_x^s, d_x^s, \ldots, d_x^s \right) \]

then according 2, then \( \beta_x = s \times a = \sum_{i=1}^{a} d_x^s \).

For example, in a circular system with 48 components, if \( j=6 \), apply Theorem 4.1.2.

1. \( a = (48/6) = 8 \) define \( d_x = (8,8,8,8,8,8) \) i.e.; \( X = \{1,9,17,25,33,41\} \Rightarrow \beta_x = 8 \).
2. Since 2 divide 6, then \( 2 \times 8 = \sum_{i=1}^{a} d_i = 16 \) take \( d_x = (1,15,1,15,1,15), (2,14,2,14,2,14), \ldots \), to
\[ (7,9,7,9,7,9) \Rightarrow \beta_x = 16 \] .
3. Since 3 divide 6, then \( 3 \times 8 = \sum_{i=1}^{a} d_i \) take \( \beta_x = 24 \).
\[ d_x = (1,8,15,1,15,8,15), (2,7,15,2,7,15), \ldots \]
\[ (7,8,9,7,8,9), (7,9,8,7,9,8) \Rightarrow \beta_x = 24 \]
\[ d_x = (1,9,14,1,9,14,1,9,14) \ldots \Rightarrow \beta_x = 24 \]

Theorem 4.1.3: Consider a circular system of \( n \) components, \( j \) the number of failed components, If \( \alpha > 1 \) is a common divisor of \( a \) and \( n \), then there exist \( X \in P(1^n) \), such that \( \frac{n}{a} = \beta_x < n \).

Proof: If \( \alpha > 1 \), and \( a \) divide \( n \), then there exists \( m \in \mathbb{Z} \), such that \( ma = n : m > 1 \). Since \( a > 1 \) also divide \( j \), let \( s = \frac{j}{a} < j \). Now, take \( d_x^s, d_x^s, \ldots, d_x^s \) any number, such that \( m = \sum_{i=1}^{a} d_x^s \) and fix any \( x_i \in 1^n \), and let \( x_2 = f_n^{d_x^s}(x_i) \), \( x_3 = f_n^{d_x^s}(x_2) = f_n^{d_x^s}(x_3) \ldots \), \( x_{s+1} = f_n^{d_x^s}(x_i) \), \( \ldots \) take \( X \) which generated by
\[ d_x = \left( d_x^s, d_x^s, \ldots, d_x^s, d_x^s, d_x^s, \ldots, d_x^s \right) \]
\[ \Rightarrow t \beta_x = t \left( \sum_{i=1}^{r} d_x^s \right) = \left( \sum_{i=1}^{r} d_x^s \right) + \left( \sum_{i=1}^{r} d_x^s \right) + \ldots + \left( \sum_{i=1}^{r} d_x^s \right) \]
\[ \Rightarrow t \times s = j \]
This implies that \( gcd(n,j) = t > 1 \), which contradicts the assumption.

5. The consecutive \( k \)-out-of-\( n \): F systems
5.1. The consecutive \( k \)-out-of-\( n \): F circular system
Let \( 1^n \) be the indices of the components in the CC(\( k,n \)), it fails if it contains any \( k \) consecutive failed components. If \( X = \{x_1 < \ldots < x_j \} \subseteq 1^n \) represents the system and consists of all indices of the failed components, then the system fails if, and only if any of the following is valid:

Note that, \( |X| = j, n = \sum_{i=1}^{j} d_x^s \), and according to 3 in Theorem 4.1.2, \( m = \beta_x < n \).

Theorem 4.1.4: Consider a circular system of \( n \) components, \( X = \{x_1 < \ldots < x_j \} \subseteq 1^n \) represents the system. If \( gcd(n,j) = 1 \) then \( \beta_x = n \).

Proof: Assume \( X = \{x_1 < \ldots < x_j \} \subseteq 1^n \) and \( \beta_x < n \).

Lemma 4.1.1. implies \( \beta_x \) divides \( n \), there exist \( t > 1 \) such that \( t \beta_x = n \). Since \( f_n^{d_x^s}(X) = X \), hence
\[ \forall x \in X \text{ there exists } x \in X \text{ such that } f_n^{d_x^s}(x) = x \neq x \text{, also since } x < x_{i+1}, \text{ then } f_n^{d_x^s}(x_{i+1}) = x_{i+1} \neq x_{i+1}. \text{ Let } d_x = \left( d_x^s, d_x^s, \ldots, d_x^s \right) \text{ the rotation of } X, \text{ WLOG take } i + s + 1 < j \], hence
\[ f_n^{d_x^s}(x_{i+1}) = f_n^{d_x^s}(x_{i+s+1}) = x_{i+s+1} \Rightarrow \beta_x = \sum_{i=1}^{j} d_x^s \text{ also } \]
\[ f_n^{d_x^s}(x_{i+1}) = f_n^{d_x^s}(x_{i+s+1}) = x_{i+s+1} \Rightarrow \beta_x = \sum_{i=1}^{j} d_x^s \text{,}
\[ t \beta_x = \sum_{i=1}^{r} d_x^s = \left( \sum_{i=1}^{r} d_x^s \right) + \left( \sum_{i=1}^{r} d_x^s \right) + \ldots + \left( \sum_{i=1}^{r} d_x^s \right) \]
\[ = \sum_{i=1}^{r} d_x^s + \sum_{i=1}^{r} d_x^s + \ldots + \sum_{i=1}^{r} d_x^s \]
\[ \Rightarrow t \times s = j \]
This implies that \( gcd(n,j) = t > 1 \) which contradicts the assumption.
1. \( \Gamma_{i-k+1} \subseteq X \), where the \((n+1)\)th is the 1st component, the \((n+2)\)th is the 2nd component ... etc.

2. \( d_k^{i} = k - 1 \) for any \( i \in I_j \), where 

\[ d_k = (d_k^{i}, d_k^{j}, \ldots, d_k^{n}) \]

is the rotation of \( X \).

Note that, if \( X \) hold any of the above conditions, then \( X \) is a “failed set”, otherwise “functioning set”, e.g. in the CC(3,7) \( X = 127, Y = 2346 \) are failed sets, since 

\[ d_{127} = (1,5,1) = \sum_{j=0}^{3} d_j^{i} = 1+1 = 3 \], and \( \Gamma_i \subseteq 2346 \), while the set \( Z = 137 \) is a functioning set. In this context, \( \Phi^k(X) \) is the ISF function of the CC(k,n)

\[ I_k(X) = \begin{cases} \bigcup_{\alpha \in \mathcal{Z}} f^\alpha (i) \subseteq X \text{ for some } i \in \Gamma_i, \\ 0 \text{ otherwise.} \end{cases} \]

Hence the failure (functioning) space of the CC(k,n) is 

\[ \Phi^k(X) = \{ X \in P(\Gamma_i) : I_k(X) = 1(0) \} \]

Note that, the symmetric property of the components in the circular system is applicable in the CC(k,n), i.e. 

\[ P(\Gamma_i) \text{ as well as } \Phi^k(X) \text{ the functioning (failure)} \]

space of the CC(k,n) are a union of mutual pair-wise disjoint classes like \( [X] \), therefore, the reliability (unreliability) function of the CC(k,n) are 

\[ R_k = \sum_i R[X] \], \( F_k = \sum_i F[X] \) respectively.

**For example**, in the i.i.d., CC(2,8), the functioning class \([1357] = [1357, 2468] \) in \( \Theta^2 \) consists of a functioning sets, and 

\[ R_2 = p_{137} + p_{2468} = 2p_\alpha^k \]

while, the failed class \([12, 23, 34, 45, 56, 67, 78, 18] \) in \( \Psi^2 \) consists of a failed sets, hence 

\[ F_2 = p_{12} + p_{23} + \ldots + p_{18} = 8p_\alpha^k \]

**Lemma 5.1.1:** Consider the CC(k,n), and \( Y \in [X] \), i.e. 

\[ f^\alpha(X) = Y \]

for some \( \alpha \in \mathcal{Z} \), then

1. If \( X \) is a functioning (failed) set, then \( Y \) also is a functioning (failed) set.
2. If the components are \( s \)-independent then 

\[ R(X) = p_{\Gamma^{L}(X)} \]

3. If the components are \( i.i.d \), then 

\[ R(X) = R(X) \] and 

\[ F(X) = F(X) \].

**Proof:**

1. Let \( X \) be a failed set, i.e. for any \( r \in \Gamma_i \)

\[ \bigcup_{\beta=0}^{i-1} f^\beta(r) \subseteq X \]

since \( f^\alpha \) is a bijection for each 

\[ \alpha \in \mathcal{Z} \], then 

\[ f^\alpha \bigcup_{\beta=0}^{i-1} f^\beta(r) \subseteq f^\alpha(X) = Y \]

implies, \( Y \) is a failed set (The same proof for the functioning set, if we put \( \alpha \) instead of \( \subseteq \)).

2. \( R(X) = R(Y) = p_r = p_{\Gamma^{L}(X)} \) and 

\[ F(X) = F(Y) = p_r = p_{\Gamma^{L}(X)} \]

3. Since \( f^\alpha \) is a bijection function for each \( \alpha \), then 

\[ [X] \subset [Y] \]

and the components are \( i.i.d \), i.e.; 

\[ p_j = p : j = 1,2, \ldots, n \]

then 

\[ R(X) = p_r = p_{\Gamma^{L}(X)} \]

\[ F(X) = F(Y) = p_r \]

(The same proof for 

\[ F(X) = F(X) \].

**5.2. The Consecutive k-out-of-n: F linear system**

Let \( \Gamma_i \) be the indicees of the components of the CL(k,n), the system fails if it contains \( k \) failed components. If \( X = \{x_1, \ldots, x_j\} \subseteq \Gamma_i \) represents the system, and consists of all indices of the failed components, then the system fails “\( X \) is a failed set for the CL(k,n)” if, and only if \( \Gamma_{i-k+1} \subseteq X \) where \( i \in \Gamma_{i-k+1} \), e.g. in the CL(2,5) \( X = 12 \) is a failed set, while \( Y = 13 \) is a functioning set.

In this context, the ISF function of the consecutive k-out-of-n: F linear system 

\[ I_k(X) = \begin{cases} 1 \text{ for some } i \in \Gamma_{i-k+1} \\ 0 \text{ otherwise.} \end{cases} \]

Hence the failure (functioning) space of the CL(k,n) is 

\[ \Phi^k(X) = \{ X \in P(\Gamma_i) : I_k(X) = 1(0) \} \]

It is worth mentioning that, many researchers consider the CC(k,n) as a generalization of the CL(k,n), due to connection between the \( i^{th} \) and the \( n^{th} \) components, this connection generates extra possible system failures, which leads to the fact \( \Psi^k \subseteq \Psi^k \), or \( \Theta^k \supseteq \Theta^k \) for any \( k \leq n \). In this context, we will omit the extra failure states from \( \Psi^k \) to achieve \( \Psi^k \).

**Lemma 5.2.1:** Consider the CC(k,n) system, and 

\[ X = \{x_1, \ldots, x_j\} \subseteq \Psi^k \]

represents the system, such that 

\[ d_k = (d_k^1, d_k^2, \ldots, d_k^n) \]

then \( X \notin \Psi^k \) or \( X \notin \Theta^k \), if \( X \) holds the following boundary conditions:

1. \( d_k^1 = 1 \)

2. \( \sum_{j=0}^{i-1} d_j^i = k - 1 \) for any \( i \in \Gamma \)

3. \( \sum_{j=0}^{i-1} d_{i-j}^i > k - 1 \) for all \( i \in \Gamma \).

**Proof:** Due to connection between the \( i^{th} \) and the \( n^{th} \) components, it is clear that the \( i^{th} \) and the \( n^{th} \) condition ensure that \( X \notin \Psi^k \), but the \( 3^{rd} \) condition, implies that, 

\[ d_i + \ldots + d_{i-k+2} > k - 1 \]

then there exist \( d_i > 1 \) for some \( i \in \Gamma \), which implies that 

\[ I_{\Gamma_{i-k+1}} \subseteq X \] for all \( i \in \Gamma \), hence \( X \notin \Psi^k \).
For example in the CC(3,8), \( d_{1278} = (1,5,1,1) \Rightarrow d_4 = 1, \) and \( d_{i} + d_{k} = 2, d_{i} + d_{k} = 2 \Rightarrow 1278 \notin \Psi_k, \) but \( \sum_{i=0}^{k-2} d_{i,j}^2 > 3 \) for all \( i \in I_1. \Rightarrow 1278 \notin \Psi_k. 


6.1. The unreliability function of the CC(k,n).

If \( j \) is the number of failed components in the CC(k,n),

1. Find \( F_j. \)
2. For each \( j=k,k+1, \ldots, n, \) determine the rotations \( (d_1,d_2, \ldots, d_n) \) such that \( n = \sum_{i=1}^{k} d_i \) and \( \sum_{i=0}^{k-2} d_{i,j}^2 = k-1 \) for some \( i \in I_1. \)

1.2. Find the corresponding failed sets \( X \in P(I_1). \)
1.3. Find the corresponding \( [X], \beta_x \) and then \( F^c. \)
1.4. If \( d_r = d^c_r, \) then \([y] = [X]. \)

2. The unreliability function \( F_c = \sum_{i=1}^{n} F^c_i. \)

Example 5.1.1: Find the failure function of the \( i.i.d. \) CC(3,8).

Find all \( (d_1,d_2,d_3) \) such that \( 8 = \sum_{i=1}^{3} d_i \) and \( \sum_{i=0}^{2} d_{i,j}^2 = 2 \) for some \( i \in I_1. \) i.e. \( d_{i,j} + d_{i+1,j} = 2 \) for some \( i \in I_1. \)

When \( j=3, \)

(1,1,6) \( \Rightarrow \{123\} = d_{123} \Rightarrow \beta_{123} = 8 \)
\([123] = [123,234,345,456,567,678,178,128] \)
\( \Rightarrow F[123] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=4, \)

(1,1,1,5) \( \Rightarrow \beta_{1234} = 8 \Rightarrow F[1234] = 8p_k \)
\([1234] = [1234,2345,3456,4567,5678,1788,678,1238] \)
\( \Rightarrow F[1234] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=5, \)

(1,1,1,1,4) \( \Rightarrow \beta_{12345} = 8 \Rightarrow F[12345] = 8p_k \)
\([12345] = [12345,23456,34567,45678,56789,67890,12345] \)
\( \Rightarrow F[12345] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=6, \)

(1,1,1,1,3) \( \Rightarrow \beta_{123456} = 8 \Rightarrow F[123456] = 8p_k \)
\([123456] = [123456,234567,345678,456789,567890,678901,123456] \)
\( \Rightarrow F[123456] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=7, \)

(1,1,1,1,1,2) \( \Rightarrow \{1234567\} \Rightarrow \beta_{1234567} = 8 \Rightarrow F[1234567] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=8, \)

(1,1,1,1,1,1) \( \Rightarrow \{12345678\} \Rightarrow \beta_{12345678} = 1 \Rightarrow R[12345678] = p_k \Rightarrow F^c = p_k \)

F_c = \sum_{i=1}^{8} F^c_i = p_k^2 + 28p_k^3 + 48p_k^4 + 32p_k^5 + 8p_k^6 

6.2. The unreliability function of the CL(k,n).

If \( j \) is the number of failed components in the CL(k,n)

1. For each \( j=2, k+1, \ldots, n, \) find all classes of \( \Psi^c \) using the above algorithm of CC(k,n)
2. Find all rotation for every class in \( \Psi^c \)
3. Use Lemma 5.2.1 to omit all corresponding \( X \) from the classes in \( \Psi^c \) such that \( X \notin \Psi^c \)
4. Compute \( F_j. \)
5. The unreliability function \( F_c = \sum_{i=1}^{n} F^c_i. \)

Example 5.2.1: Find the failure function of the \( i.i.d. \) CL(3,8).

If \( j \) is the number of failed components, omit all states, such that, \( d_i + d_j = 2 \) or \( d_{i+1} + d_j = 2, \) and \( d_i + d_{i+1} \neq 2: i = 1, \ldots, j-2, \)

When \( j=3, \)

(1,1,6) \( \Rightarrow \{123\} \Rightarrow \beta_{123} = 8 \Rightarrow F[123] = 6p_k \Rightarrow F^c = 6p_k \)

When \( j=4, \)

(1,1,1,5) \( \Rightarrow \beta_{1234} = 8 \Rightarrow F[1234] = 6p_k \Rightarrow F^c = 6p_k \)

When \( j=5, \)

(1,1,1,1,4) \( \Rightarrow \beta_{12345} = 8 \Rightarrow F[12345] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=6, \)

(1,1,1,1,3) \( \Rightarrow \beta_{123456} = 8 \Rightarrow F[123456] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=7, \)

(1,1,1,1,1,2) \( \Rightarrow \{1234567\} \Rightarrow \beta_{1234567} = 8 \Rightarrow F[1234567] = 8p_k \Rightarrow F^c = 8p_k \)

When \( j=8, \)

(1,1,1,1,1,1) \( \Rightarrow \{12345678\} \Rightarrow \beta_{12345678} = 1 \Rightarrow R[12345678] = p_k \Rightarrow F^c = p_k \)

F_c = \sum_{i=1}^{8} F^c_i = p_k^2 + 28p_k^3 + 48p_k^4 + 32p_k^5 + 8p_k^6 

6
\[(1,1,3,1,2)(2,1,1,3,1)(1,2,1,1,3)(3,1,2,1,1)(1,3,2,1,1)\]
\[\Rightarrow F[12367] = 6p_k^8\]
\[F_i^c = 41p_k^3\]

When \(j=6\),
\[(1,1,1,1,1,3) \Rightarrow \{12345\} \Rightarrow F[12345] = 8p_k^2\]
\[(1,1,1,1,2,2) \Rightarrow \{12345\} \Rightarrow F[12345] = 8p_k^2\]
\[(1,1,1,2,1,2) \Rightarrow \{1,2,4,2,1,1\} \Rightarrow F[123467] = 7p_k^2\]
\[(1,1,2,1,1,2) \Rightarrow F[123567] = 4p_k^2 \Rightarrow F_i^c = 27p_k^8\]

When \(j=7\),
\[(1,1,1,1,1,1,2) \Rightarrow F[1234567] = 8p_k^1 \Rightarrow F_i^c = 8p_k^6\]

When \(j=8\),
\[(1,1,1,1,1,1,1) \Rightarrow R[12345678] = p_k^0 \Rightarrow F_i^c = p_k^0.\]

\[F_L = \sum_{i=1}^{k} F_i^c = p_k^0 + 8p_k^4 + 27p_k^8 + 41p_k^3 + 25p_k^4 + 6p_k^5\]

6.3. The reliability function of the CC(k,n).
If \(j\) is the number of failed components in the CC(k,n).
1. Find \(R_j^c\).
1.1. If \(j = 0,1,\ldots,k-1\), then all states are functioning sets, then \(R_j^c = \left(\begin{array}{c} n \\ j \end{array}\right)p_k^{n-j}\)

1.2. If \(M\) is the maximum total number of the failed components that allows the system to function [10], then
\[M = \left\{n-\left[(n/k)\right], \; n \text{ is multiple of } k\right\}
\[n-1-\left[(n/k)\right], \; n \text{ is not multiple of } k\]

For \(j=k+1,\ldots,M\), find the rotations \(d_1, d_2,\ldots,d_j\) hold the boundary conditions \(\sum_{i=1}^{j}d_i > j-1\) for all \(i \in I_j\), and \(n = \sum_{i=1}^{j}d_i\).

1.3. Find the corresponding functioning sets \(X \in P\{1\}\)

1.4. Find the class \([X]\), \(\beta_X\) and then \(R_j^c\).

2. The reliability, \(R_c = \sum_{j=0}^{M} R_j^c\)

Example 6.1.1: Find the reliability function of the \(i.i.d.\) CC(3,8).
\[M = 8-\left\{\left[8/3\right]\right\} = 5.\]
The boundary conditions \(d_i + d_{i+j} > j-1\) for all \(i \in I_j\), and \(8 = \sum_{i=1}^{j}d_i\).

When \(j=0\), \(\left\{\begin{array}{c} 8 \\ 0 \end{array}\right\} \Rightarrow R_0^c = p_k^8\), the only class \([\varnothing]\]

When \(j=1\), \(\left\{\begin{array}{c} 8 \\ 1 \end{array}\right\} \Rightarrow R_1^c = 8p_k^6\), the only class \([1]\)

When \(j=2\), \(\left\{\begin{array}{c} 8 \\ 2 \end{array}\right\} \Rightarrow R_2^c = 28p_k^6\), the classes are \([12],[13],[14],[15]\)

When \(j=3\),
\[(1,2,5) \Rightarrow \{124\} \Rightarrow \beta_{24} = 8 \Rightarrow R[124] = 8p_k^6\]
\[(1,3,4) \Rightarrow \{125\} \Rightarrow \beta_{25} = 8 \Rightarrow R[125] = 8p_k^6\]
\[(1,4,3) \Rightarrow \{126\} \Rightarrow \beta_{26} = 8 \Rightarrow R[126] = 8p_k^6\]
\[(1,5,2) \Rightarrow \{127\} \Rightarrow \beta_{27} = 8 \Rightarrow R[127] = 8p_k^6\]
\[(2,2,4) \Rightarrow \{135\} \Rightarrow \beta_{135} = 8 \Rightarrow R[135] = 8p_k^6\]
\[(2,3,3) \Rightarrow \{136\} \Rightarrow \beta_{136} = 8 \Rightarrow R[136] = 8p_k^6\]
\[R_i^c = 48p_k^3\]

When \(j=4\),
\[(1,2,1,4) \Rightarrow \{1245\} \Rightarrow \beta_{1245} = 8 \Rightarrow R[1245] = 8p_k^4\]
\[(1,2,2,3) \Rightarrow \{1256\} \Rightarrow \beta_{1256} = 8 \Rightarrow R[1256] = 8p_k^4\]
\[(1,2,3,2) \Rightarrow \{1247\} \Rightarrow \beta_{1247} = 8 \Rightarrow R[1247] = 8p_k^4\]
\[(1,3,1,3) \Rightarrow \{1256\} \Rightarrow \beta_{1256} = 4 \Rightarrow R[1256] = 4p_k^4\]
\[(1,3,2,2) \Rightarrow \{1257\} \Rightarrow \beta_{1257} = 8 \Rightarrow R[1257] = 8p_k^4\]
\[(2,2,2,2) \Rightarrow \{1357\} \Rightarrow \beta_{1357} = 2 \Rightarrow R[1357] = 2p_k^4\]
\[R_i^c = 38p_k^3\]

When \(j=5\),
\[(1,2,1,2,2) \Rightarrow X = \{12457\} \Rightarrow \beta_{12457} = 8 \Rightarrow R[12457] = 8p_k^4\]
\[R_i^c = \sum_{j=0}^{M} R_j^c = p_k^8 + 8p_k^7 + 28p_k^6 + 48p_k^5 + 38p_k^4 + 8p_k^3\]

6.4. The reliability function of the CL(k,n).
If \(j\) is the number of failed components in the CL(k,n), apply the same above algorithm, but add the omitted failed sets from \(\Psi^c\) using lemma 5.1.2, and add them to \(\Theta_i\) according the value of \(j=k,k+1,\ldots,M\).

7. Conclusion
In this paper, new algorithms to find the reliability and unreliability functions of the consecutive \(k\)-out-of-\(n\): F linear and circular systems are obtained. The failure and the functioning space of the circular consecutive \(k\)-out-of-\(n\): F system is classified according the parameters \(k\) and \(n\) into finite pairwise disjoint classes. For the linear consecutive system, boundary conditions are specified to omit some failure states from the failure space of the circular case to achieve the failure space of the linear case, which facilitate the finding the reliability and unreliability functions of the linear case.

References
[1]. Antonpoulou, I. and Papastavridis, S., “Fast recursive algorithm to evaluate the reliability of a circular consecutive \(k\)-out-of-\(n\): F


