# New Algorithms to Find Reliability and Unreliability Functions of the Consecutive *k*-out-of-*n*: F Linear & Circular System

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**Abstract:** The consecutive *k*-out-of-*n*: F linear and circular system consists of *n* components. The system fails if at least *k* consecutive components are in the failure state. In this paper, new algorithms to find the reliability and unreliability functions of the consecutive *k*-out-of-*n*: F linear and circular system are obtained, in this context, the Index Structure Function (ISF), and two equivalence relations are defined to partition the failure and the functioning space of the consecutive *k*-out-of-*n*: F circular system into finite mutual pairwise disjoint classes respectively, where the reliability and unreliability functions are the summations of reliability and unreliability of these equivalence classes. For the linear consecutive *k*-out-of-*n*: F system, a boundary conditions are given to omit a specified failure states from the failure space of the circular case to achieve the failure states of the linear case, which facilitate the evaluation of the reliability and unreliability functions of the consecutive *k*-out-of-*n*: F linear system.

**Keywords:** Consecutive *k*-out-of-*n*: F systems, System reliability, equivalence relation, modular arithmetic.

**Notations**:

L(C) : Linear (Circular)

*i.i.d.* : Independent and Identically distributed

mod : Modular arithmetic function

gcd : greatest common devisor

$\mathbb{I}_j^i$ : $\{i, i+1, ..., j\}$

$P(\mathbb{I}_n^1)$ : The power set of $\mathbb{I}_n^1$

$X=\{x_1<...<x_j\}$: a set $X=\{x_1,x_2,...,x_j\}$ such that $x_i<x_h$ for all $i<h$

$i\underset{j}{\oplus}s$ $=(i+s)\bmod j$, if $i+s=nj \Rightarrow i\underset{j}{\oplus}s=j$

$p_i(q_i)$ :The reliability (unreliability) of the $i^{th}$ component.

$\overline{X}$ : The complement of the set *X*.

$p_X$ $=\prod_{i\in X}p_i\prod_{j\in X}q_j$

$p_n^s$ $=p(n,s)=p^{n-s}q^s$

$R(X)=p_X$ , $F(X)=p_X$

$\mathbb{R}[X]$ : Reliability of the class $[X]$

$\mathbb{F}[X]$ : Unreliability of the class $[X]$

C(*k*,*n*) : Consecutive *k*-out-of-*n*: F system.

CL(*k*,*n*) : Consecutive *k*-out-of-*n*: F linear system.

CC(*k*,*n*) : Consecutive *k*-out-of-*n*: F circular system.

$|X|$ : The cardinality of the set *X*.

$f^\alpha$ : The composite function $\alpha$ times.

$\mathbb{R}_j^{L(C)},\mathbb{F}_j^{L(C)}$ : The reliability, unreliability function of the consecutive *k*-out-of-*n*: F linear (circular) when the number of failed components is *j*.

## 1. Introduction

Through the last 2 decades in the last century, many researchers studied intensively and extensively the reliability of the consecutive *k*-out-of-*n*: F system "C(*k*,*n*)" due its importance in applications, low expense and high reliability, the system consists of *n* components, the components are ordered sequentially in a line or a circle, the system fails if, and only if at least *k* consecutive components are in the failure state. Chiang and Niu [5] provided the two famous applications (Telecommunication System with *n* Relay Stations (either satellites or ground stations) and Pipeline of Transmit Oil System with *n* Pumps), Microwave Stations of a Telecom Network [4], Vacuum System in an Electron Accelerator and Photographing of a Nuclear Accelerator [9].

Chiang & Niu [5] obtained the first recursive algorithm for computing the reliability of the consecutive *k*-out-of-*n*: F linear system "CL(*k*,*n*)", Bollinger [2-3] introduced a simple and direct combinatorial formula for calculating the failure function of the system, while Derman et al. [6] was the first one who introduced and calculated the reliability of the consecutive *k*-out-of-*n*: F circular system "CC(*k*,*n*)", they provided a recursive algorithm for computing the reliability of the CL(*k*,*n*) and CC(*k*,*n*) with *i.i.d.* components, Shanthikumar [13] computed the reliability of the CL(*k*,*n*), when the components are stochastically independent with unequal failure probabilities, Hwang [7] studied both the CL(*k*,*n*) & CC(*k*,*n*), two recursive algorithms were introduced using two arguments for the CL(*k*,*n*), he extended also the argument of Deman et al. [6] to introduce a recursive equation for the CC(*k*,*n*). Lambiris and Papastavridis [10] introduced the first exact formula for the CL(*k*,*n*) and CC(*k*,*n*), where the components are *i.i.d.*, Antonopoulou and Papastavridis [1] studied only the CC(*k*,*n*); they introduced a faster recursive algorithm than Hwang [7] and all other previous algorithms that obtaining the reliability of the stochastically independent components with fewer components, Wu & Chen [15] generalized the system through adding some conditions and restriction , and transformed the CC($k$,$n$) to the CL($k$,$n$) by adding virtual components to the circular system $n$+1, $n$+2, …, $n$+$i$, …., $n$+$k$-1, where $n$+$i$=$i$, . Many recursive algorithms, lower and upper bounds, optimal design and exact formula were introduced by [8-9], [11], [12] and [14].

In this paper new algorithms to find the reliability and unreliability functions of the CC($k$,$n$) & CL($k$,$n$) are obtained. We recall and adapt the needed facts, and results from the set theory that is concerning relation between components in the circular system, the ISF and equivalence relations are defined, these relations are extended to represent the failure and functioning space of the CC($k$,$n$) & CL($k$,$n$), which yields to compute the reliability and unreliability functions.

The following assumptions are assumed to be satisfied by all systems below:

1. The state of the component and the system is either " functioning" or "failed"
2. All the components are mutually statistically independent.

## 2. The Index Structure Function

**Definition 2.1**: Consider a system consists of $n$ stochastically independent components, $\mathbb{I}_n^1$ is the indices of the components. Define the Index Structure Function (ISF) of the $i^{\text{th}}$ component

$$\Gamma(i)=\begin{cases}0, & \text{if the } i^{th} \text{ component functions}\\ 1, & \text{if the } i^{th} \text{ component fails}\end{cases}$$

The set $X=\{i\in\mathbb{I}_n^1:\Gamma(i)=1\}\subseteq\mathbb{I}_n^1$, represents the system and consists of all indices of the failed components e.g., the set $X=\{1,2\}$, for simply, $X=12$ indicates that the 1st and the 2nd components are only the failed components. Define the ISF of the whole system, $I:P(\mathbb{I}_n^1)\to\{0,1\}$,

$$I(X)=\begin{cases}0, & \text{if the system is functioning}\\ 1, & \text{if the system is failed}\end{cases}$$

Where, $P(\mathbb{I}_n^1)$ is the failure space of the components. The ISF is concerning the components positions and indices that is determining whether the system fails or not.

**Definition 2.2**: $X\subseteq\mathbb{I}_n^1$ is called functioning set, if $I(X)=0$, vice versa, a failed set if $I(X)=1$. In this context, the ISF divides $P(\mathbb{I}_n^1)$ into two main following disjoint sub collections:

- $\Theta=\{X\in P(\mathbb{I}_n^1):I(X)=0\}$ the functioning space of the system
- $\Psi=\{Y\in P(\mathbb{I}_n^1):I(Y)=1\}$ the failed space of the system.

## 3. Reliability Using the ISF

**Definition 3.1**: The reliability of the $i^{\text{th}}$ component is $P(\Gamma(i)=0)=p_i$, hence if the system represented by the functioning set $X\in\Theta$, then the reliability of the system is $R(X)=P(I(X)=0)=p_X$, which implies, the reliability function of the system $\mathbb{R}$ is the probability that the system has any functioning set; i.e., $\mathbb{R}=\sum_{X\in\Theta}R(X)$.

While, the unreliability of the $i^{\text{th}}$ component is $P(\Gamma(i)=1)=q_i=1-p_i$. If the system represented by the failed set $Y\in\Psi$, then the system unreliability is $F(Y)=P(I(Y)=1)=p_Y$ and the unreliability function of the system $\mathbb{F}$ is the probability that the system has any failed set $\mathbb{F}=\sum_{Y\in\Psi}F(Y)=1-\mathbb{R}$.

## 4. The Circular System Model

The circular system model is any system consists of $n$ stochastically independent components, and arranged in a circle. Define the circular system model using ISF, where $\mathbb{I}_n^1$ is the indices of the components, and the set $X\subseteq\mathbb{I}_n^1$ represents the system, which consists of all failed components.

The positions of the components in the circle applies that the participation of any component in the system status, "whether functioning or failed" is equivalent to the participation of any other component. To simulate this property, define the bijection function, $f_n:\mathbb{I}_n^1\to\mathbb{I}_n^1$ $f_n(x)=(x \bmod n)+1$ for any $x\in\mathbb{I}_n^1$, and a relation between any $X,Y\in P(\mathbb{I}_n^1)$ such that $X\equiv Y$ if, and only if, there exist $\alpha\in\mathbb{Z}$ such that $f_n^{\alpha}(X)=Y$.

**Lemma 4.1**: $(\equiv)$ is an equivalence relation.

**Proof**: **Reflexivity**: $X\sim X$ since $X=f_n^{n}(X)$

**Transitivity**:If $X\sim Y \Leftrightarrow$ there exists $\alpha_1$ such that $Y=f_n^{\alpha_1}(X)$ and $Y\sim Z\Leftrightarrow$ there exists $\alpha_2$ such that $Z=f_n^{\alpha_2}(Y)\Rightarrow Z=f_n^{\alpha_2}(f_n^{\alpha_1}(X))=f_n^{\alpha_1+\alpha_2}(X)\Rightarrow X\sim Z$.

**Symmetry**: If $X\sim Y$ there exists $\alpha$ such that $Y=f_n^{\alpha}(X)$, hence $X=f_n^{n-\alpha}(Y)\Leftrightarrow Y\sim X$.

According to lemma 4.1 $P(\mathbb{I}_n^1)$ is a union of a finite partition of mutually disjoint classes like $[X]=\{f_n^{\alpha}(X):\alpha=1,2,...,n\}$, as well as the $\Theta$, and $\Psi$.

**For example**, in the circular system of 6 components, $[13]=\{13,24,35,46,15,26\}$.

### 4.1. The Rotations in the Circular System

**Definition 4.1.1**: Let $X\in P(\mathbb{I}_n^1)$ represents the circular system, define the orbit of $X$ as $\beta_X=\min\{\alpha\in\mathbb{Z}: f_n^\alpha(X)=X\}$. Note that $[X]=\{f_n^\alpha(X):\alpha=1,2,\dots,\beta_X\}$.

**Lemma 4.1.1:** For all $X\in P(\mathbb{I}_n^1)$, $\beta_X$ divides *n*.

**Proof**: Since $f_n^n(X)=X$ for all $X\in P(\mathbb{I}_n^1)$, then $\beta_X\le n$. If $\beta_X=n$ we done. Otherwise if $\beta_X<n$, and $\beta_X$ does not divide *n*, then $n=a\beta_X+b$, where $0<b=n \bmod \beta_X$.

$X=f_n^n(X)=f_n^{a\beta_X+b}(X)=f_n^b\left(f_n^{a\beta_X}(X)\right)=f_n^b(X)$ but, $b<\beta_X$, which contradicts that $\beta_X$ is the minimum integer such that $X=f_n^{\beta_X}(X)$.

**Note:** If $Y\in[X]$

1. There exist $\alpha\in\mathbb{Z}$ such that $f_n^\alpha(X)=Y$, but $f_n^{\beta_X}(Y)=f_n^{\beta_X}(f_n^\alpha(X))=f_n^\alpha(f_n^{\beta_X}(X))=f_n^\alpha(X)=Y$ $\Rightarrow\beta_Y=\beta_X$
2. $|X|=|Y|$.
3. $\mathbb{R}[X]=\sum_{Z\in[X]}R(Z)=\sum_{Z\in[X]}p_Z=\sum_{\alpha=1}^{\beta_X}p_{f_n^\alpha(X)}$.
4. $\mathbb{F}[X]=\sum_{Z\in[X]}F(Z)=\sum_{Z\in[X]}p_Z=\sum_{\alpha=1}^{\beta_X}p_{f_n^\alpha(X)}$.

**Definition 4.1.2** : Let $X=\{x_1<\dots<x_j\}\subseteq\mathbb{I}_n^1$, represents the circular system, define $d_X=(d_1^X,d_2^X,\dots,d_j^X)$ the rotation of *X*, where $d_i^X\ge1$ is the minimum integer number such that $f_n^{d_i^X}(x_i)=x_{i+1}$, for *i*=1,2,…, *j*-1, and $f_n^{d_j^X}(x_j)=x_1$, also define the following:

1. $d_X^r=(d_{j-r+1}^X,\dots,d_j^X,d_1^X,\dots,d_{j-r}^X)$ is the $r^{th}$ rotation of $X$. (note that $d_X^{j+s}=d_X^s$ ).
2. The set of all rotations of the set $X$ is $D(X)=\{d_X^r: r=1,2,\dots,j\}$.

**Note**: If $X=\{x_1<\dots<x_j\}\subseteq\mathbb{I}_n^1$, then, anyone can easily shows that

1. $n=\sum_{i=1}^j d_i^X$.
2. If $d_i^X=t$ for all $i=1,2,\dots,j$, then $\beta_X=t$.

**Definition 4.1.3:** Let $X,Y\in P(\mathbb{I}_n^1)$ be any states of the circular system, we say " $Y$ is a rotation of *X*", denoted by $X\sim Y$ if $d_Y\in D(X)$ or there exists $r\in\{1,2,\dots,j\}$ such that $d_Y=d_X^r$.

**Lemma 4.1.3**: ( ~ ) is an equivalence relation.

**Proof: Reflexivity**: $X\sim X\Leftrightarrow d_X^j=d_X$

**Transitivity**: Let $X\sim Y\Leftrightarrow$ there exist $r_1$ such that $d_Y=d_X^{r_1}$, and if $Y\sim Z\Leftrightarrow$ there exist $r_2$ such that $d_Z=d_Y^{r_2}$, then $d_Z=d_Y^{r_2}=d_X^{r_2\oplus_j r_1}$

**Symmetry**: $X\sim Y\Leftrightarrow$ there exist $r$ such that $d_Y=d_X^r\Rightarrow$
$d_X=d_Y^{j-r}\Rightarrow Y\sim X$.

**Theorem 4.1.1:** $X\equiv Y\Leftrightarrow X\sim Y$.

**Proof**: Assume $X\equiv Y\Rightarrow$ there exist $\alpha$ such that $f_n^\alpha(X)=Y$, $f_n^\alpha$ is a bijection function for each $\alpha$, hence, $|X|=|Y|=j$. let $Y=\bigcup_{i=1}^j f_n^\alpha(x_i)$, **WLOG**, let $f_n^\alpha(x_i)=y_i$ for all $i=1,2,\dots,j$, if $d_X=(d_1^X,d_2^X,\dots,d_j^X)$, then $f_n^{d_i^X}(y_i)=f_n^{d_i^X}(f_n^\alpha(x_i))=f_n^\alpha(f_n^{d_i^X}(x_i))=$ $=f_n^\alpha(x_{i+1})=y_{i+1}\Rightarrow d_X^i=d_Y^i\Leftrightarrow d_X=d_Y^j$ then $X\sim Y$.

**Conversely**: if $X\sim Y$, there exist $r\in\{1,\dots,j\}$ such that, $d_Y=d_X^r$. Since $|X|=|Y|=j$, let $X=\bigcup_{t=1}^j x_t, Y=\bigcup_{t=1}^j y_t$, as a result of $r^{th}$ rotation, the element $x_t$ will rotate to the element $y_{f_j^r(t)}$, define the function $g(x_t)=y_{f_j^r(t)}$, let $\alpha=\min\{\gamma\in\mathbb{Z}: f_n^\gamma(x_t)=g(x_t),\forall x_t\in X\}$, if and only if $f_n^\alpha(X)=Y\Leftrightarrow X\equiv Y$.

**For example** in the circular system with 10 components, if $X=\{149\},Y=\{168\}\in P(\mathbb{I}_{10}^1)\Rightarrow$ $d_X=(3,5,2),d_Y=(5,2,3)\Rightarrow d_Y=d_X^2\Rightarrow g(x_t)=y_{f_j^2(t)}$ $\Rightarrow g(x_1)=g(1)=y_3=8, g(x_2)=g(4)=y_1=1,$ $g(x_3)=g(9)=y_2=6\Rightarrow$ $\alpha=\min\{\gamma\in\mathbb{Z}: f_{10}^\gamma(x_t)=g(x_t):\forall x_t\in X\}$ $=\min\{7,17,\dots\}=7\Rightarrow f_{10}^7(\{149\})=\{168\}$.

**Note:** According Theorem 4.1.1 if $d_Y=d_X^r\Rightarrow$ $[X]=[Y]$.

**Theorem 4.1.2**: Consider a circular system of n components, *j* the number of failed components, then

1. If $a\in\mathbb{Z}$ such that $a\times j=n$, then there exist $X_a\in P(\mathbb{I}_n^1)$ such that $|X_a|=j$ and $\beta_{X_a}=a$.
2. If $X\in P(\mathbb{I}_n^1)$ such that $|X|=j$ and

$$d_X=\left(\overbrace{\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{1},\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{2},\ldots\ldots,\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{r}}^{j}\right),$$ then $\beta_X=\sum_{i=1}^{s}d_i^X$.

3. If $s,a\in\mathbb{Z}$ such that $a\times j=n$, and $s$ divide $j$, then there exist $Y_s\in P(\mathbb{I}_n^1)$ and $|Y_s|=j$ such that $\beta_{Y_s}=s\times a$.

**Proof:**

1. Take $d_{X_a}=\left(\underbrace{a,\ldots,a}_{j\text{ times}}\right)$, fix any $x_i\in\mathbb{I}_n^1$, and $x_{i+1}=f_n^a(x_i)\ :i\in\mathbb{I}_{j-1}^1$, hence $f_n^a(x_j)=x_1$. Define $X_a\in P(\mathbb{I}_n^1)$ such that $X_a=\bigcup_{i=1}^{j}x_i$, then $f_n^a(X_a)=f_n^a\left(\bigcup_{i=1}^{j}x_i\right)=\bigcup_{i=1}^{j}f_n^a(x_i)=\bigcup_{i=0}^{j-1}x_{i+1}=X_a$ $\Leftrightarrow\beta_{X_a}=a$.

2. Use theorem 4.1.1 $d_X=d_X^s$
$$\alpha=\min\left\{\gamma\in\mathbb{Z}:f_n^\gamma(x_t)=x_{f_j^s(t)}\right\}=\sum_{i=1}^{s}d_i^X=\beta_X$$

3. Use 1, there exist $X_a\in P(\mathbb{I}_n^1)$ with $\beta_{X_a}=a$, where $d_{X_a}=\left(\underbrace{a,a,\ldots,a}_{j\text{ times}}\right)$, take $d_1^X,d_2^X,\ldots,d_s^X$ not all of them $a$, such that $sa=\sum_{i=1}^{s}d_i^X$, define the set $Y_s$ such that
$$d_{Y_s}=\left(\overbrace{\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{1},\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{2},\ldots\ldots,\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{a=(j/s)}}^{j}\right)$$
then according 2, then $\beta_{Y_s}=s\times a=\sum_{i=1}^{s}d_i^X$.

**For example,** in a circular system with 48 components, if $j$=6, apply Theorem 4.1.2.

1. $a=(48/6)=8$ define $d_{X_a}=(8,8,8,8,8,8)$ i.e.; $X_a=\{1,9,17,25,33,41\}\Rightarrow\beta_{X_a}=8$.
2. Since 2 divide 6, then $2\times8=\sum_{i=1}^{2}d_i=16$ take $d_{Y_s}=(1,15,1,15,1,15),(2,14,2,14,2,14),\ldots,$ to $(7,9,7,9,7,9)\Rightarrow\beta_{Y_s}=16$.
3. Since 3 divide 6, then $3\times8=\sum_{i=1}^{3}d_i$, take $d_{Y_{s_i}}=(1,8,15,1,8,15),(2,7,15,2,7,15),\ldots,$ $(7,8,9,7,8,9),(7,9,8,7,9,8)\Rightarrow\beta_{Y_{s_i}}=24$
$d_{Z_s}=(1,9,14,1,9,14)\ldots\Rightarrow\beta_{Z_s}=24$

**Theorem 4.1.3:** Consider a circular system of $n$ components, $j$ the number of failed components, If $a>1$ is a common divisor of $j$ and $n$, then there exist $X\in P(\mathbb{I}_n^1)$, such that $\frac{n}{a}=\beta_X<n$.

**Proof**: If $a>1$, and $a$ divide $n$, then there exists $m\in\mathbb{Z}$, such that $ma=n:m>1$. Since $a>1$ also divide $j$, let $s=\frac{j}{a}<j$. Now, take $d_1^X,d_2^X,\ldots,d_s^X$ any number, such that $m=\sum_{i=1}^{s}d_i^X$ and fix any $x_1\in\mathbb{I}_n^1$, and let $x_2=f_n^{d_1^X}(x_1)$,
$x_3=f_n^{d_2^X}(x_2)=f_n^{\sum_{i=1}^{2}d_i^X}(x_1)\ldots,x_{s+1}=f_n^{\sum_{i=1}^{s}d_i^X}(x_1)=f_n^{d_s^X}(x_s)$ take $X$ which generated by
$$d_X=\left(\overbrace{\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{m=\beta_X=\sum_{i=1}^{s}d_i^X},\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{m=\beta_X=\sum_{i=1}^{s}d_i^X},\ldots,\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{m=\beta_X=\sum_{i=1}^{s}d_i^X}}^{a\text{ times}}\right).$$
Note that, $|X|=j$, $n=\sum_{i=1}^{j}d_i^X$, and according to 3 in Theorem 4.1.2, $m=\beta_X<n$.

**Theorem 4.1.4**: Consider a circular system of $n$ components, $X=\{x_1<\ldots<x_j\}\subseteq\mathbb{I}_n^1$ represents the system. If gcd($n$,$j$)=1 then $\beta_X=n$.

**Proof**: Assume $X=\{x_1<\ldots<x_j\}\subseteq\mathbb{I}_n^1$ and $\beta_X<n$, Lemma 4.1.1. implies $\beta_X$ divides $n$, there exist $t>1$ such that $t\beta_X=n$. Since $f_n^{\beta_X}(X)=X$, hence $\forall x_i\in X$ there exists $x_{i\oplus_j s}\in X$ such that $f_n^{\beta_X}(x_i)=x_{i\oplus_j s}\neq x_i$, also since $x_i<x_{i+1}$, then $f_n^{\beta_X}(x_{i+1})=x_{i+1\oplus_j s}\neq x_{i+1}$. Let $d_X=(d_1^X,d_2^X,\ldots,d_j^X)$ the rotation of $X$, WLOG take $i+s+1<j$, hence
$$f_n^{\beta_X}(x_i)=f_n^{\sum_{r=0}^{s-1}d_{i+r}^X}(x_i)=x_{i+s}\Rightarrow\beta_X=\sum_{r=0}^{s-1}d_{i+r}^X,\text{ also}$$
$$f_n^{d_{i+1}^X}(x_{i+1})=x_{i+2},f_n^{\sum_{r=1}^{2}d_{i+r}^X}(x_{i+1})=x_{i+3},\ldots,f_n^{\sum_{r=1}^{s}d_{i+r}^X}(x_{i+1})=x_{i+1+s}\Rightarrow$$
but $f_n^{\beta_X}(x_{i+1})=x_{i+1+s}\Rightarrow\beta_X=\sum_{r=1}^{s}d_{i+r}^X\Rightarrow d_i^X=d_{i+s}^X$

Since $t\beta_X=n=\sum_{i=1}^{n}d_i^X$, then
$$d_X=\left(\overbrace{\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{\beta_X},\underbrace{d_1^X,d_2^X,\ldots,d_s^X}_{\beta_X},\ldots\ldots,\underbrace{d_{j-s+1}^X,d_{j-s+2}^X,\ldots,d_j^X}_{\beta_X}}^{t\text{ times}}\right)$$

$$\left.\begin{array}{l} d_1^X = d_{s+1}^X = d_{2s+1}^X = \ldots = d_{j-s+1}^X \\ d_2^X = d_{s+2}^X = d_{2s+2}^X = \ldots = d_{j-s+2}^X \\ \vdots \\ d_s^X = d_{2s}^X = d_{3s}^X = \ldots = d_{j-s+s}^X \end{array}\right\}$$

$$t\beta_X = t\left(\sum_{i=1}^{s} d_i^X\right) = \underbrace{\left(\sum_{i=1}^{s} d_i^X\right) + \left(\sum_{i=1}^{s} d_i^X\right) + \ldots + \left(\sum_{i=1}^{s} d_i^X\right)}_{t \text{ times}}$$

$$= \underbrace{\left(\sum_{i=1}^{s} d_i^X\right) + \left(\sum_{i=s+1}^{2s} d_i^X\right) + \left(\sum_{i=2s+1}^{3s} d_i^X\right) + \ldots + \left(\sum_{i=j-s+1}^{j} d_i^X\right)}_{(j/s)=t \text{ times}}$$

$$\Rightarrow t \times s = j$$

This implies that $\gcd(n,j) = t > 1$ which contradicts the assumption.

# 5. The consecutive *k*-out-of-*n*: F systems

## 5.1. The consecutive *k*-out-of-*n*: F circular system

Let $\mathbb{I}_n^1$ be the indices of the components in the CC(*k*,*n*), it fails if it contains any *k* consecutive failed components. If $X = \{x_1 < \ldots < x_j\} \subseteq \mathbb{I}_n^1$ represents the system and consists of all indices of the failed components, then the system fails if, and only if any of the following is valid:

1. $\mathbb{I}_{i+k-1}^i \subseteq X, i \in \mathbb{I}_n^1$ where the $(n+1)^{\text{th}}$ is the $1^{\text{st}}$ component, the $(n+2)^{\text{th}}$ is the $2^{\text{nd}}$ component …etc.)
2. $\sum_{s=0}^{k-2} d_{i \oplus_j s}^X = k-1$ for any $i \in \mathbb{I}_j^i$, where $d_X = (d_1^X, d_2^X, \ldots, d_j^X)$ is the rotation of *X*.

Note that, if *X* hold any of the above conditions, then *X* is a "failed set", otherwise "functioning set", e.g. in the CC(3,7) $X = 127, Y = 2346$ are failed sets, since $d_{127} = (1,5,1) \Rightarrow \sum_{s=0}^{3-2} d_{3 \oplus_j s}^X = 1+1 = 3-1 = 2$, and $\mathbb{I}_4^2 \subseteq 2346$, while the set $Z = 137$ is a functioning set. In this context, the ISF function of the CC(*k*,*n*)

$$I_C(X) = \begin{cases} 1 & \bigcup_{\alpha=0}^{k-1} f_n^\alpha(i) \subseteq X \quad \text{for some } i \in \mathbb{I}_n^1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \left\{\begin{array}{ll} 0 & \sum_{s=0}^{k-2} d_{i \oplus_j s}^X > k-1, \text{for all } i \in \mathbb{I}_j^1 \\ 1 & \sum_{s=0}^{k-2} d_{i \oplus_j s}^X = k-1, \text{for some } i \in \mathbb{I}_j^1 \end{array}\right\}$$

Hence the failure (functioning) space of the CC(*k*,*n*) is $\Psi_C^k(\Theta_C^k) = \{X \in P(\mathbb{I}_n^1) : I_C(X) = 1(0)\}$.

Note that, the symmetric property of the components in the circular system is applicable in the CC(*k*,*n*), i.e. $P(\mathbb{I}_n^1)$ as well as, $\Theta_C^k(\Psi_C^k)$ the functioning (failure) space of the CC(*k*,*n*) are a union of mutual pairwise disjoint classes like $[X]$, therefore, the reliability (unreliability) function of the CC(*k*,*n*) are $\mathbb{R}_C = \sum_{[X] \in \Theta_C^k} \mathbb{R}[X]$, $\mathbb{F}_C = \sum_{[X] \in \Psi_C^k} \mathbb{F}[X]$ respectively.

**For example,** in the *i.i.d.*, CC(2,8), the functioning class $[1357] = \{1357, 2468\} \in \Theta_C^2$ consists of a functioning sets, and $\mathbb{R}_C[1357] = p_{1357} + p_{2468} = 2p_8^4$, while, the failed class $[12] = \{12, 23, 34, 45, 56, 67, 78, 18\} \in \Psi_C^2$ consists of a failed sets, hence $\mathbb{F}_C[12] = p_{12} + p_{23} + \ldots + p_{18} = 8p_8^6$.

**Lemma 5.1.1**: Consider the CC(*k*,*n*), and $Y \in [X]$, i.e. $f_n^\alpha(X) = Y$, for some $\alpha \in \mathbb{Z}$, then

1. If *X* is a functioning (failed) set, then *Y* also is a functioning (failed) set.
2. If the components are *s*-independent then $\mathbb{R}_C(Y) = p_{f_n^\alpha(X)}, \mathbb{F}_C(Y) = p_{f_n^\alpha(X)}$
3. If the components are *i.i.d.*, then $\mathbb{R}_C(Y) = \mathbb{R}_C(X)$ and $\mathbb{F}_C(Y) = \mathbb{F}_C(X)$.

**Proof**:

1. Let *X* be a failed set, i.e. for any $r \in \mathbb{I}_n^1$ $\bigcup_{\beta=0}^{k-1} f^\beta(r) \subseteq X$, since $f_n^\alpha$ is a bijection for each $\alpha \in \mathbb{Z}$, then $f_n^\alpha\left(\bigcup_{\beta=0}^{k-1} f^\beta(r)\right) \subseteq f_n^\alpha(X) = Y$, which implies, *Y* is a failed set (The same proof for the functioning set, if we put $\not\subset$ instead of $\subseteq$).
2. $\mathbb{R}_C(Y) = R(Y) = p_Y = p_{f_n^\alpha(X)}$ and $\mathbb{F}_C(Y) = R(Y) = p_Y = p_{f_n^\alpha(X)}$
3. Since $f_n^\alpha$ is a bijection function for each $\alpha$, then $|X| = |Y|$, and if the components are *i.i.d.*, i.e.; $p_j = p : j = 1, 2, \ldots, n$, then $\mathbb{R}_C(Y) = p_Y = p_n^{|Y|} = $ $= p_n^{|X|} = p_X = \mathbb{R}_C(X)$ (The same proof for $\mathbb{F}_C(Y) = \mathbb{F}_C(X)$).

## 5.2. The Consecutive *k*-out-of-*n*: F linear system

Let $\mathbb{I}_n^1$ be the indices of the components of the CL(*k*,*n*), the system fails if it contains *k* failed components. If $X = \{x_1 < \ldots < x_j\} \subseteq \mathbb{I}_n^1$ represents the system, and consists of all indices of the failed components, then the system fails "*X* is a failed set for the CL(*k*,*n*)" if, and only if $\mathbb{I}_{i+k-1}^i \subseteq X$ where $i \in \mathbb{I}_{n-k+1}^1$, e.g. in the CL(2,5) $X = 12$ is a failed set, while $Y = 13$ is a functioning set.

In this context, the ISF function of the consecutive $k$-out-of-$n$: F linear system

$$I_L(X)=\begin{cases}1 & \mathbb{I}^i_{i+k-1}\subseteq X \text{ for some } i\in\mathbb{I}^1_{n-k+1}\\ 0 & \text{otherwise.}\end{cases}$$

Hence the failure (functioning) space of the CL($k$,$n$) is $\Psi^k_L(\Theta^k_L)=\{X\in P(\mathbb{I}^1_n): I_L(X)=1(0)\}$.

It is worth mentioning that, many researchers consider the CC($k$,$n$) as a generalization of the CL($k$,$n$), due to connection between the 1st and the $n$th components, this connection generates extra possible system failures, which leads to the fact $\Psi^k_L\subseteq\Psi^k_C$, or $\Theta^k_L\supseteq\Theta^k_C$ for any $k\le n$. In this context, we will omit the extra failure states from $\Psi^k_C$ to achieve $\Psi^k_L$.

**Lemma 5.2.1**: Consider the CC($k$,$n$) system, and $X=\{x_1<\ldots<x_j\}\in\Psi^k_C$ represents the system, such that $d_X=(d^X_1,d^X_2,\ldots,d^X_j)$, then $X\notin\Psi^k_L$ or $X\in\Theta^k_L$, if $X$ holds the following boundary conditions:

1. $d^X_j=1$,
2. $\sum_{s=0}^{k-2} d^X_{i\oplus_j s}=k-1$ for any $i\in\mathbb{I}^{j-(k-2)}_j$.
3. $\sum_{t=0}^{k-2} d^X_{i+t}>k-1$ for all $i\in\mathbb{I}^1_{j-(k-1)}$.

**Proof:** Due to connection between the 1st and the $n$th components, it is clear that the 1st and the 2nd condition ensure that $X\in\Psi^k_C$, but the 3rd condition, implies that, $d_i+\ldots+d_{i+k-2}>k-1$ for all $i\in\mathbb{I}^1_{j-(k-1)}$, then there exist $d_t>1$ for some $t\in\mathbb{I}^1_{i+k-2}$, which implies that $\mathbb{I}^i_{i+k-1}\not\subset X$ for all $i\in\mathbb{I}^1_{n-k+1}$, hence $X\notin\Psi^k_L$.

**For example** in the CC(3,8), $d_{1278}=(1,5,1,1)\Rightarrow d_4=1$, and $d_3+d_4=2, d_4+d_1=2\Rightarrow 1278\in\Psi^k_C$, but $\sum_{t=0}^{1} d^X_{i+t}>3$ for all $i\in\mathbb{I}^1_2\Rightarrow 1278\notin\Psi^k_L$.

# 6. The Proposed Algorithms.

## 6.1. The unreliability function of the CC($k$,$n$).

If $j$ is the number of failed components in the CC($k$,$n$),

1. Find $\mathbb{F}_j$.

1.1. For each $j=k,k+1,\ldots,n$, determine the rotations $(d_1,d_2,\ldots,d_j)$ such that $n=\sum_{i=1}^{j} d_i$ and $\sum_{s=0}^{k-2} d^X_{i\oplus_j s}=k-1$ for some $i\in\mathbb{I}^1_j$.

1.2. Find the corresponding failed sets $X\in P(\mathbb{I}^1_n)$.

1.3. Find the corresponding $[X]$, $\beta_X$ and then $\mathbb{F}^C_j$

1.4. If $d_Y=d^r_X$, then $[Y]=[X]$.

2. The unreliability function $\mathbb{F}_C=\sum_{j=k}^{n}\mathbb{F}^C_j$.

**Example 5.1.1**: Find the failure function of the *i.i.d.* CC(3,8).

Find all $(d_1,d_2\ldots,d_j)$ such that $8=\sum_{i=1}^{j} d_i$ and $\sum_{s=0}^{1} d_{i\oplus_j s}=2$, for some $i\in\mathbb{I}^1_j$ i.e. $d_{i\oplus_j s}+d_{i+1\oplus_j s}=2$ for some $i\in\mathbb{I}^1_j$.

**When $j$=3,**

$$(1,1,6)\Rightarrow\{123\}=d_{123}\Rightarrow\beta_{123}=8$$
$$[123]=\{123,234,345,456,567,678,178,128\}$$
$$\Rightarrow\mathbb{F}[123]=8p^5_8 \qquad \mathbb{F}^C_3=8p^5_8$$

**When $j$=4,**

$$(1,1,1,5)\Rightarrow\beta_{1234}=8\Rightarrow\mathbb{F}[1234]=8p^4_8$$
$$[1234]=\{1234,2345,3456,4567,5678,1678,1278,1238\}$$
$$(1,1,2,4)\Rightarrow\{1235\}\Rightarrow\beta_{1235}=8\Rightarrow\mathbb{F}[1235]=8p^4_8$$
$$(1,1,3,3)\Rightarrow\{1236\}\Rightarrow\beta_{1236}=8\Rightarrow\mathbb{F}[1236]=8p^4_8$$
$$(1,1,4,2)\Rightarrow\{1237\}\Rightarrow\beta_{1237}=8\Rightarrow\mathbb{F}[1237]=8p^4_8$$
$$\mathbb{F}^C_4=32p^4_8$$

**When $j$=5,**

$$(1,1,1,1,4)\Rightarrow\{12345\}\Rightarrow\beta_{12345}=8\Rightarrow\mathbb{F}[12345]=8p^3_8$$
$$(1,1,1,2,3)\Rightarrow\{12346\}\Rightarrow\beta_{12346}=8\Rightarrow\mathbb{F}[12346]=8p^3_8$$
$$(1,1,1,3,2)\Rightarrow\{12347\}\Rightarrow\beta_{12347}=8\Rightarrow\mathbb{F}[12347]=8p^3_8$$
$$(1,1,2,1,3)\Rightarrow\{12356\}\Rightarrow\beta_{12356}=8\Rightarrow\mathbb{F}[12356]=8p^3_8$$
$$(1,1,2,2,2)\Rightarrow\{12357\}\Rightarrow\beta_{12357}=8\Rightarrow\mathbb{F}[12357]=8p^3_8$$
$$(1,1,3,1,2)\Rightarrow\{12367\}\Rightarrow\beta_{12367}=8\Rightarrow\mathbb{F}[12367]=8p^3_8$$
$$\mathbb{F}^C_5=48p^3_8$$

**When $j$=6,**

$$(1,1,1,1,1,3)\Rightarrow\{123456\}\Rightarrow\beta_{123456}=8\Rightarrow\mathbb{F}[123456]=8p^2_8$$
$$(1,1,1,1,2,2)\Rightarrow\{123457\}\Rightarrow\beta_{123457}=8\Rightarrow\mathbb{F}[123457]=8p^2_8$$
$$(1,1,1,2,1,2)\Rightarrow\{123467\}\Rightarrow\beta_{123467}=8\Rightarrow\mathbb{F}[123467]=8p^2_8$$
$$(1,1,2,1,1,2)\Rightarrow\{123567\}\Rightarrow\beta_{123567}=4\Rightarrow\mathbb{F}[123567]=4p^2_8$$
$$\mathbb{F}^C_6=28p^2_8$$

**When $j$=7,**

$$(1,1,1,1,1,1,2)\Rightarrow\{1234567\}\Rightarrow\beta_{1234567}=8\Rightarrow$$
$$\mathbb{F}[1234567]=8p^1_8\Rightarrow\mathbb{F}^C_7=8p^1_8$$

**When $j$=8,**

$$(1,1,1,1,1,1,1,1)\Rightarrow\{12345678\}\Rightarrow\beta_{12345678}=1\Rightarrow$$
$$\mathbb{R}[12345678]=p^0_8\Rightarrow\mathbb{F}^C_8=p^0_8.$$

$$\mathbb{F}_C=\sum_{j=k}^{n}\mathbb{F}^C_j=p^0_8+8p^1_8+28p^2_8+48p^3_8+32p^4_8+8p^5_8$$

## 6.2. The unreliability function of the CL($k$,$n$).

If $j$ is the number of failed components in the CL($k$,$n$)

1. For each $j=k$, $k+1$, …, $n$, find all classes of $\Psi^k_C$ using the above algorithm of CC($k$,$n$)
2. Find all rotation for every class in $\Psi^k_C$.

3. Use lemma 5.2.1 to omit all corresponding $X$ from the classes in $\Psi_C^k$ such that $X \notin \Psi_L^k$,
4. Compute $\mathbb{F}_j$.
5. The unreliability function $\mathbb{F}_L = \sum_{j=k}^{n} \mathbb{F}_j$.

**Example 5.2.1:** Find the failure function of the *i.i.d.* CL(3,8).

If $j$ the number of failed components, **omit** all states, such that, $d_1 + d_j = 2$ or $d_{j-1} + d_j = 2$, and $d_i + d_{i+1} \neq 2 : i = 1,.., j-2$,

**When *j*=3,**

$$(1,1,6)\ \cancel{(6,1,1)}\ \cancel{(1,6,1)} \Rightarrow$$

$$[123] = \{123, 234, 345, 456, 567, 678, \cancel{178}, \cancel{128}\}$$

$$\mathbb{F}[123] = 6p_8^5 \qquad \mathbb{F}_3^L = 6p_8^5$$

**When *j*=4,**

$$(1,1,1,5)(5,1,1,1)\ \cancel{(1,5,1,1)}\ (1,1,5,1) \Rightarrow$$

$$[1234] = \{1234, 2345, 3456, 4567, 5678, 1678, \cancel{1278}, 1238\}$$

$$\Rightarrow \mathbb{F}[1234] = 7p_8^4$$

$$(1,1,2,4)(4,1,1,2)\ \cancel{(2,4,1,1)}\ \cancel{(1,2,4,1)} \Rightarrow \mathbb{F}[1235] = 6p_8^4$$

$$(1,1,3,3)(3,1,1,3)\ \cancel{(3,3,1,1)}\ \cancel{(1,3,3,1)} \Rightarrow \mathbb{F}[1236] = 6p_8^4$$

$$(1,1,4,2)(2,1,1,4)\ \cancel{(4,2,1,1)}\ \cancel{(1,4,2,1)} \Rightarrow \mathbb{F}[1237] = 6p_8^4$$

$$\mathbb{F}_4^L = 25p_8^4$$

**When *j*=5,**

$$(1,1,1,1,4) \Rightarrow \{12345\} \Rightarrow \mathbb{F}[12345] = 8p_8^3$$

$$(1,1,1,2,3) \Rightarrow \{12346\} \Rightarrow \mathbb{F}[12346] = 8p_8^3$$

$$(1,1,1,3,2)(2,1,1,1,3)(3,2,1,1,1)\ \cancel{(1,3,2,1,1)}\ (1,1,3,2,1)$$

$$\Rightarrow \mathbb{F}[12347] = 7p_8^3$$

$$(1,1,2,1,3)(3,1,1,2,1)(1,3,1,1,2)\ \cancel{(2,1,3,1,1)}\ \cancel{(1,2,1,3,1)}$$

$$\Rightarrow \mathbb{F}[12356] = 6p_8^3$$

$$(1,1,2,2,2)(2,1,1,2,2)(2,2,1,1,2)\ \cancel{(2,2,2,1,1)}\ \cancel{(1,2,2,2,1)}$$

$$\Rightarrow \mathbb{F}[12357] = 6p_8^3$$

$$(1,1,3,1,2)(2,1,1,3,1)(1,2,1,1,3)\ \cancel{(3,1,2,1,1)}\ \cancel{(1,3,1,2,1)}$$

$$\Rightarrow \mathbb{F}[12367] = 6p_8^3$$

$$\mathbb{F}_5^L = 41p_8^3$$

**When *j*=6,**

$$(1,1,1,1,1,3) \Rightarrow \{123456\} \Rightarrow \mathbb{F}[123456] = 8p_8^2$$

$$(1,1,1,1,2,2) \Rightarrow \{123457\} \Rightarrow \mathbb{F}[123457] = 8p_8^2$$

$$(1,1,1,2,1,2)\ \cancel{(1,2,1,2,1,1)} \Rightarrow \mathbb{F}[123467] = 7p_8^2$$

$$(1,1,2,1,1,2) \Rightarrow \mathbb{F}[123567] = 4p_8^2 \Rightarrow \mathbb{F}_6^L = 27p_8^2$$

**When *j*=7,**

$$(1,1,1,1,1,1,2) \Rightarrow \mathbb{F}[1234567] = 8p_8^1 \Rightarrow \mathbb{F}_7^L = 8p_8^1$$

**When *j*=8,**

$$(1,1,1,1,1,1,1) \Rightarrow \mathbb{R}[12345678] = p_8^0 \Rightarrow \mathbb{F}_8^L = p_8^0.$$

$$\mathbb{F}_L = \sum_{j=k}^{n} \mathbb{F}_j^L = p_8^0 + 8p_8^1 + 27p_8^2 + 41p_8^3 + 25p_8^4 + 6p_8^5$$

## 6.3. The reliability function of the CC(*k*,*n*).

If $j$ is the number of failed components in the CC($k$,$n$)

1. Find $\mathbb{R}_j^C$.

1.1. If $j = 0,1,...,k-1$, then all states are functioning sets, then $\mathbb{R}_j^C = \binom{n}{j} p_n^{n-j}$

1.2. If $M$ is the maximum total number of the failed components that allows the system to function [10], then

$$M = \begin{cases} n - [(n/k)], & n \text{ is multiple of } k \\ n-1-[(n/k)], & n \text{ is not multiple of } k \end{cases}$$

For $j$=$k$,$k$+1…,$M$, find the rotations $(d_1, d_2, ...., d_j)$ hold the boundary conditions $\sum_{t=0}^{k-2} d_{i \oplus_j t} > k-1$ for all $i \in \mathbb{I}_j^1$, and $n = \sum_{i=1}^{j} d_i$.

1.3. Find the corresponding functioning sets $X \in P(\mathbb{I}_n^1)$

1.4. Find the class $[X]$, $\beta_X$ and then $\mathbb{R}_j^C$.

2. The reliability, $\mathbb{R}_C = \sum_{j=0}^{M} \mathbb{R}_j^C$

**Example 6.1.1:** Find the reliability function of the *i.i.d.* CC(3,8).

$M = 8-1-[8/3] = 5$. The boundary conditions $d_i + d_{i \oplus_j 1} > 2$ for all $i \in \mathbb{I}_j^1$ and $8 = \sum_{i=1}^{j} d_i$.

**When *j*=0,** $\binom{8}{0} = 1 \Rightarrow \mathbb{R}_0^C = p_8^8$, the only class $[\varnothing]$

**When *j*=1,** $\binom{8}{1} = 8 \Rightarrow \mathbb{R}_1^C = 8p_8^6$, the only class $[1]$

**When *j*=2,** $\binom{8}{2} = 28 \Rightarrow \mathbb{R}_2^C = 28p_8^6$, the classes are $[12],[13],[14],[15]$

**When *j*=3,**

$$(1,2,5) \Rightarrow \{124\} \Rightarrow \beta_{124} = 8 \Rightarrow \mathbb{R}[124] = 8p_8^5$$

$$(1,3,4) \Rightarrow \{125\} \Rightarrow \beta_{125} = 8 \Rightarrow \mathbb{R}[125] = 8p_8^5$$

$$(1,4,3) \Rightarrow \{126\} \Rightarrow \beta_{126} = 8 \Rightarrow \mathbb{R}[126] = 8p_8^5$$

$$(1,5,2) \Rightarrow \{127\} \Rightarrow \beta_{127} = 8 \Rightarrow \mathbb{R}[127] = 8p_8^5$$
$$(2,2,4) \Rightarrow \{135\} \Rightarrow \beta_{135} = 8 \Rightarrow \mathbb{R}[135] = 8p_8^5$$
$$(2,3,3) \Rightarrow \{136\} \Rightarrow \beta_{136} = 8 \Rightarrow \mathbb{R}[136] = 8p_8^5$$
$$\mathbb{R}_3^C = 48p_8^5$$

**When *j*=4,**

$$(1,2,1,4) \Rightarrow \{1245\} \Rightarrow \beta_{1245} = 8 \Rightarrow \mathbb{R}[1245] = 8p_8^4$$
$$(1,2,2,3) \Rightarrow \{1246\} \Rightarrow \beta_{1246} = 8 \Rightarrow \mathbb{R}[1246] = 8p_8^4$$
$$(1,2,3,2) \Rightarrow \{1247\} \Rightarrow \beta_{1247} = 8 \Rightarrow \mathbb{R}[1247] = 8p_8^4$$
$$(1,3,1,3) \Rightarrow \{1256\} \Rightarrow \beta_{1256} = 4 \Rightarrow \mathbb{R}[1256] = 4p_8^4$$
$$(1,3,2,2) \Rightarrow \{1257\} \Rightarrow \beta_{1257} = 8 \Rightarrow \mathbb{R}[1257] = 8p_8^4$$
$$(2,2,2,2) \Rightarrow \{1357\} \Rightarrow \beta_{1357} = 2 \Rightarrow \mathbb{R}[1357] = 2p_8^4$$
$$\mathbb{R}_4^C = 38p_8^4$$

**When *j*=5,**

$$(1,2,1,2,2) \Rightarrow X = \{12457\} \Rightarrow \beta_{12457} = 8$$
$$\Rightarrow \mathbb{R}[12457] = 8p^3q^5 \Rightarrow \mathbb{R}_5^C = 8p_8^3$$

The reliability function is

$$\mathbb{R}_C = \sum_{j=0}^{M} \mathbb{R}_j^C = p_8^8 + 8p_8^7 + 28p_8^6 + 48p_8^5 + 38p_8^4 + 8p_8^3$$

### 6.4. The reliability function of the CL(*k*,*n*).

If *j* is the number of failed components in the CL(*k*,*n*), apply the same above algorithm, but add the omitted failed sets from $\Psi_C^k$ using lemma 5.1.2, and add them to $\Theta_L^k$ according the value of $j=k,k+1,\ldots M$.

## 7. Conclusion

In this paper, new algorithms to find the reliability and unreliability functions of the consecutive *k*-out-of-*n*: F linear and circular systems are obtained. The failure and the functioning space of the circular consecutive *k*-out-of-*n*: F system is classified according the parameters *k* and *n* into finite pairwise disjoint classes. For the linear consecutive system, boundary conditions are specified to omit some failure states from the failure space of the circular case to achieve the failure space of the linear case, which facilitate the finding the reliability and unreliability functions of the linear case.

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